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Fuel-efficient navigation in complex flows

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***Abstract*— In realistic time-chaotic flows, the time-dependent separatrices are revealed by Lagrangian coherent structures (LCS). The LCS surfaces organize complex flows, revealing dynamical channels useful for weakly propelled mobile agents. We propose a feedback control strategy to explicitly incorporate LCS, which are in general evolving. Inanc et al. (2005) observed that the fuel optimal trajectory in ocean flow seems to lie on a moving LCS surface. We take this observation a step further by explicitly incorporating the moving LCS location into the control strategy. We have the vehicle track an LCS surface, considering this a time-dependent boundary following problem, with the aim of following lanes of fuel-efficient motion delineated by LCS surfaces. We demonstrate this strategy in a double-gyre flow and find that indeed a strategy using LCS is feasible and uses less fuel than a naive direct targeting approach.**

I. INTRODUCTION

Inanc et al. [2], [6] observed that the fuel optimal trajectory for a fully actuated point vehicle in ocean surface flow seems to lie on a moving Lagrangian coherent structure (LCS), which in this case is a moving curve. LCS are the time-dependent analogs of stable and unstable manifolds [1], [4] which have been shown in other contexts to delineate lanes of fuel-efficient travel [3], [5]. Inspired by the observation of Inanc et al., we take a next logical step and develop this observation into a new control algorithm by explicitly including information regarding the LCS location (and therefore the lanes of fuel-efficient travel) into our control algorithm.

We suppose that optimal trajectory generation within a moving medium can be divided into three main regimes. Considering as a system parameter the available amount of control, we define:

$$U_{ratio} = \frac{U_{max}}{max\{V(x,y,t)\}} \tag{1}$$

where $V(x,y;t)$ is the velocity field and U_{max} is the amount of velocity which can be produced by the control.

For $U_{ratio} \gg 1$, one can ignore the velocity field; for $U_{ratio} \leq 0.5$ the available control is too weak and the dominant transport process is natural advection. An interesting situation arises when $0.5 \leq U_{ratio} \leq 1$. In this regime the control—together with the exploitation of the velocity field surrounding the vehicle—can produce quasi-passive motion optimizing the fuel consumption. Since this regime has not been adequately explored in previous studies, we focus our work on this U_{ratio} range.

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We consider a double gyre flow as an exemplar of the types of flow of interest. The 2D velocity field $V(x,y,t) = (u(x,y,t), v(x,y,t))$ of the double gyre is defined analytically as:

$$u = -\pi A \sin(\pi f(x)) \cos(\pi y) \tag{2}$$

$$v = \pi A \cos(\pi f(x)) \sin(\pi y) \frac{df}{dx} \tag{3}$$

where $f(x,t) = a(t)x^2 + b(t)x$, $a(t) = \epsilon \sin(\omega t)$, $b(t) = 1 - 2\epsilon \sin(\omega t)$, and where A is related to the amplitude of the velocity, ϵ measures the amplitude of the oscillation of the gyre (i.e. the shifting of the saddle $(1,0)$ during the motion) and ω is the frequency of the system. We study this flow over the domain $[0,2] \times [0,1]$ and we fix $A = 0.1$, $\epsilon = 0.1$ and $\omega = \frac{2\pi}{10}$.

The computation of LCS requires knowledge of the velocity field for a future finite time. In real applications this can be done by forecasting future conditions. In our case, we have the analytical expression of the velocity field. The use of LCS is of great interest because they provide clear overall information about the characteristics of the flow which are not obvious from the observation of the velocity field. For a stationary velocity field the LCS are invariant manifolds of, e.g., fixed hyperbolic points, but since the system is non-autonomous the LCS are time-dependent.

Choosing a starting point and target point, we analyze different control strategies considering as parameters the time taken to reach the target and the control cost. In the trivial case of these two points laying exactly on an LCS the optimal trajectory is represented by the LCS itself and no control action is required (as seen in Inanc et al. [2], [6]).

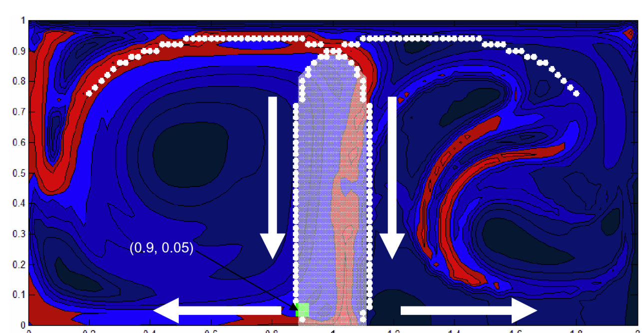

Fig. 1. This plot shows the position of an LCS (white dots) for two instants of time. The instants of time were chosen in order to have the LCS on the extreme position (on the left and on the right of $(1,0)$). The white arrows represent the direction of the flow while the green square is the target point. The LCS fluctuate during the motion but they always stays in the shaded region (recall $\epsilon = 0.1$).

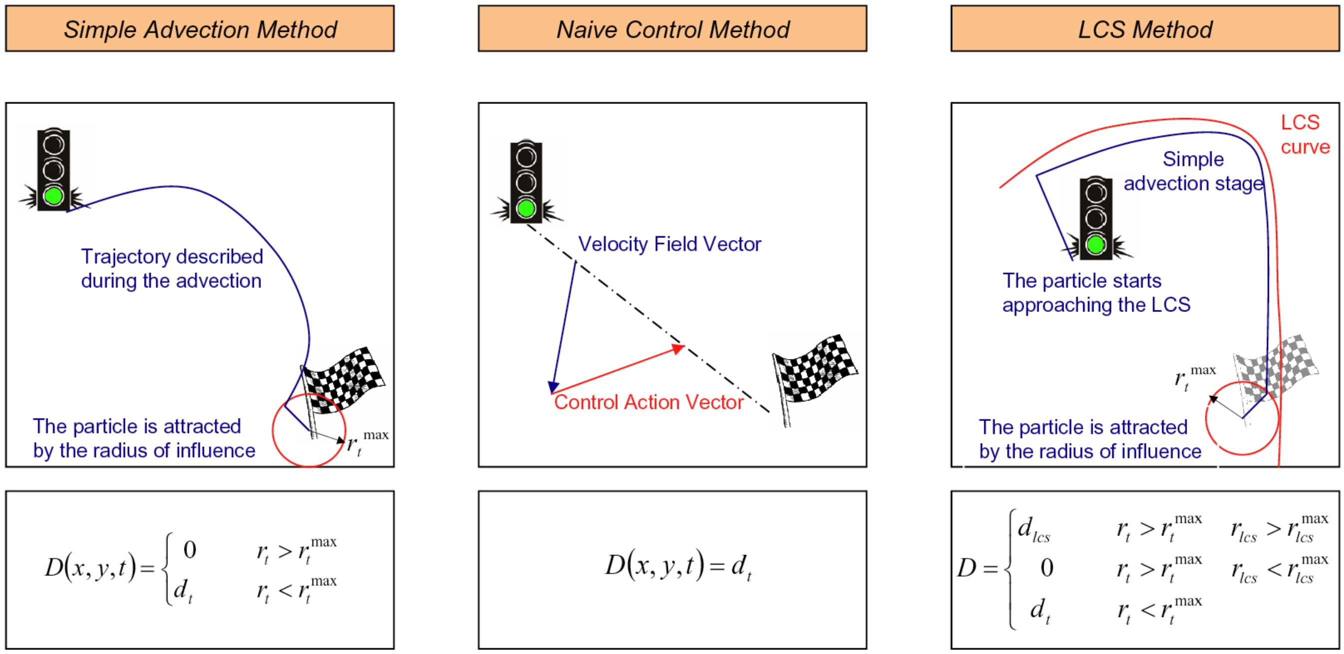

Fig. 2. Schematic representation of the control strategies which are used. $D(x, y, t)$ is defined in a different way for each method. r_t and r_{lcs} are the distance of the particle from the target and the LCS respectively while r_t^{max} and r_{lcs}^{max} are the radius of influence of the target and the LCS. d_t and d_{lcs} are unit vectors pointing to the target and the LCS respectively.

We consider the nontrivial case. We fix the point $(0.9, 0.05)$ as target site. As shown in Fig. 1 the lower boundary of the domain surrounding the point $(1, 0)$ behaves as a moving saddle point: the flow is attracted along the vertical direction and repelled along the horizontal one. This guarantees that the LCS always stays in the vicinity of this point providing an optimal path to follow in order to reach $(0.9, 0.05)$.

The following equations of motion incorporating the control law $D(x, y, t)$ are used:

$$\dot{x} = V_{flow} + V_{control} = V(x, y, t) + U_{max} \cdot D(x, y, t) \quad (4)$$

where D is a vector which provides the direction for the U_{max} control action. For this preliminary study we did two hypothesis: first we treated the vehicle as a massless particle without considering the interaction between the particle and the flow; second we supposed U_{max} to be constant and so the control was acting as input of velocity.

II. CONTROL STRATEGIES

The three control strategies considered will be labeled as the Simple Advection method (SA from now on), the Naive Control method (NC from now on) and the LCS method. In Fig. 2 we show the logic of the three strategies.

In the SA method we simply drop the particle and observe where it is advected. We define a certain radius of influence of the target r_t^{max} and, if the particle comes within this distance from the target, some control action is taken (via the NC strategy) to drive the particle to the target. Note that when $D(x, y, t) = 0$ there is no control action and so the particle is simply advected (no fuel consumption).

For the NC method, we calculate at each instant of time the direction D to the target. We orient our control action in order to obtain a net velocity in the direction of the target. In this case the control action is always present and the control tries to reach the target by augmenting the velocity such that the resultant velocity is pointing toward the target.

In the LCS method, we track an LCS surface, considering it a time-dependent boundary following problem with the aim of following lanes of efficient motion. We first give some control input to the particle in order to jump onto the closest LCS curve. When the particle is close to the LCS we turn the control off and let the particle be advected by the flow. When the particle eventually comes close enough to the target point we again activate the NC control in order to jump on it.

From the description of the three methods it is possible to make some preliminary observations. Clearly the SA method works only if we drop the particle on an LCS or reasonably close to it; the NC method is effective but needs more energy because the control is always active; the LCS method should combine the good properties of the former two providing lower fuel consumption together with a reasonable time of convergence (by convergence we mean the accomplishment of the given task, i.e., reaching the target).

III. LCS CALCULATION

The LCS are defined as the ridge of the maximum Finite Time Lyapunov Exponent (FTLE from now on) [1], [7]. The FTLE (the maximum is understood) can be calculated as:

$$\sigma_{t_0}^T(x) = \frac{1}{|T|} \ln \left\| \frac{d\phi_{t_0}^{t_0+T}(x)}{dx} \right\| \quad (5)$$

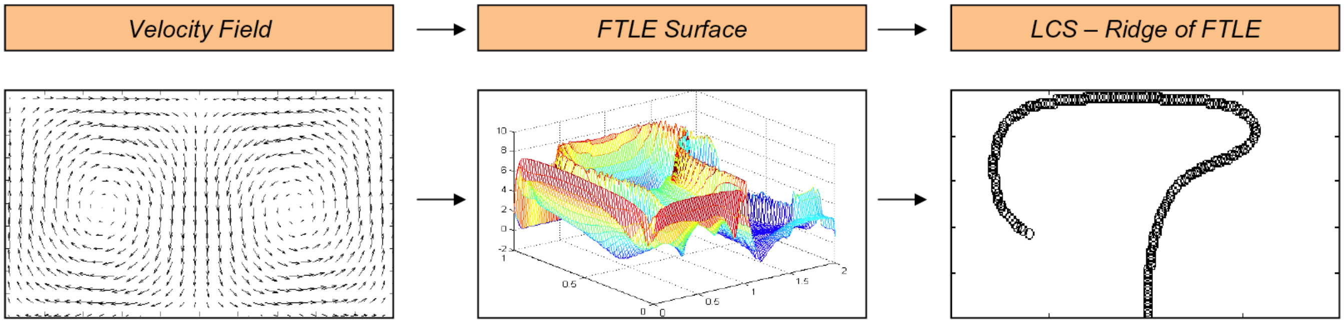

Fig. 3. Steps needed to calculate LCS curves.

where $\phi_{t_0}^{t+T}(x)$ is the flow map which maps the particle from its position $x(t_0) = x$ at time t_0 to the position $x(t) = \phi_{t_0}^{t+T}(x)$ at time $t_0 + T$. The norm in eq. (5) is $\|A\| = \sqrt{\lambda_{\max}(A^*A)}$, where A^* is the transpose of the square matrix A and $\lambda_{\max}(M)$ denotes the maximum eigenvalue of the matrix M.

The FTLE measures the maximum linearized growth rate (over a duration T) of the distance between particle trajectories starting near a reference particle trajectory at time t_0. If an (infinitesimal) circular blob of particles is placed about the reference trajectory at time t_0, then after a duration $T = t - t_0$, the blob will have expanded in some directions and compressed in others to form an ellipse.

In order to calculate the FTLE field, or spatial distribution of FTLE, we use the software package MANGEN[1]. MANGEN calculates the FTLE from an analytically or data-defined velocity field. The FTLE field is a scalar function $\sigma_{t_0}^{T}(x, y)$ over a domain of initial conditions in the 2D plane (i.e., $(x, y) \in \Omega \subset \mathbb{R}^2$). The FTLE can be represented as a surface in 3D space, $(x, y, \sigma_{t_0}^{T}(x, y))$.

Ridges of high $\sigma_{t_0}^{T}(x)$ correspond to LCS surfaces [7] of dimension 1, i.e., LCS curves. In general, for all 2D flows, including time-chaotic ones, LCS are repelling curves for $T > 0$ and attracting curves for $T < 0$. In our study, we consider repelling curves because evidence suggests [2] that repelling curves provide the energy efficient transport alleyways.

In order to obtain the LCS curve from the FTLE field we need to extract the ridge from the FTLE. This was accomplished using a simple x-y scanning procedure. We cut the FTLE in slices along the x and y direction and we look for local maxima in these slices. Combining the information coming from all the slices we are able to reconstruct the LCS curve. The process of obtaining the LCS curve is summarized in Fig. 3.

IV. RESULTS

Running numerical simulations we collected data regarding the LCS and NC methods (SA method was of scarce interest) for different values of U_{ratio} in the domain $[0.5, 1.5]$. We show an example of the three control strategies in Fig. 4. The results have been summarized in Control Cost vs Time plots (Figs. 5 and 6) where data for same initial condition x_0 and varying U_{ratio} were plotted together. The control cost has been defined as

$$Control\ Cost = U_{max} \cdot \Delta t \tag{6}$$

where Δt is the total time the control is switched on.

Analyzing the results some conclusions can be drawn. For small U_{ratio} the NC method does not work because the velocity field is too strong and the control is not able to force the particle in the right direction. The NC method results usually lay on a straight line. For small U_{ratio}

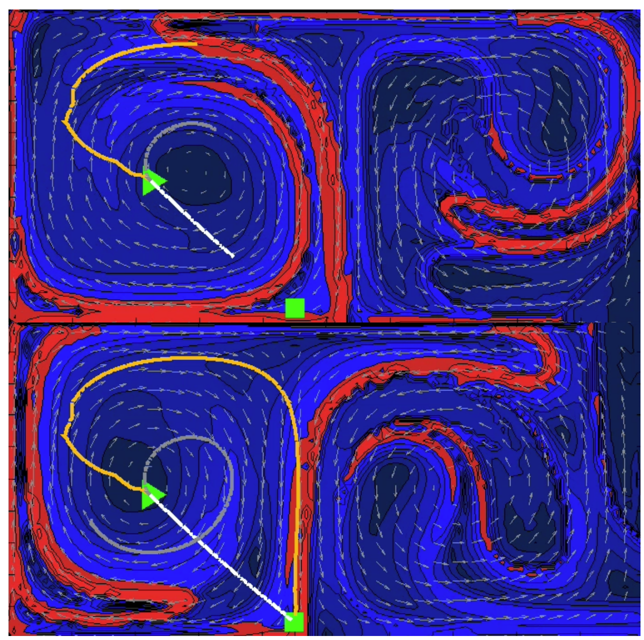

Fig. 4. This figure shows the paths of three vehicles in a double-gyre flow. The gray one is passively advected by the flow (SA method), while the orange (LCS method) and white (NC method) vehicles are attempting to go from a starting point (green triangle) to a target (green square), by turning on or off an engine capable of producing a speed comparable to the maximum speed of the flow itself, U_{max}. Under these conditions, the white vehicle makes it to the target first, but uses more than 10 times the fuel of the orange vehicle, which cleverly navigates using Lagrangian coherent structures (LCS, red features). The non-intuitive initial behavior of the fuel-efficient orange vehicle is a consequence of the global properties of the flow. The vehicles are released at the same time and are shown at an intermediate (top panel) and final time (bottom panel). The full movie of the motion can be found on the web at `www.esm.vt.edu/~sdross/movies/methods.mov`

[1]`http://www.mangen.info`

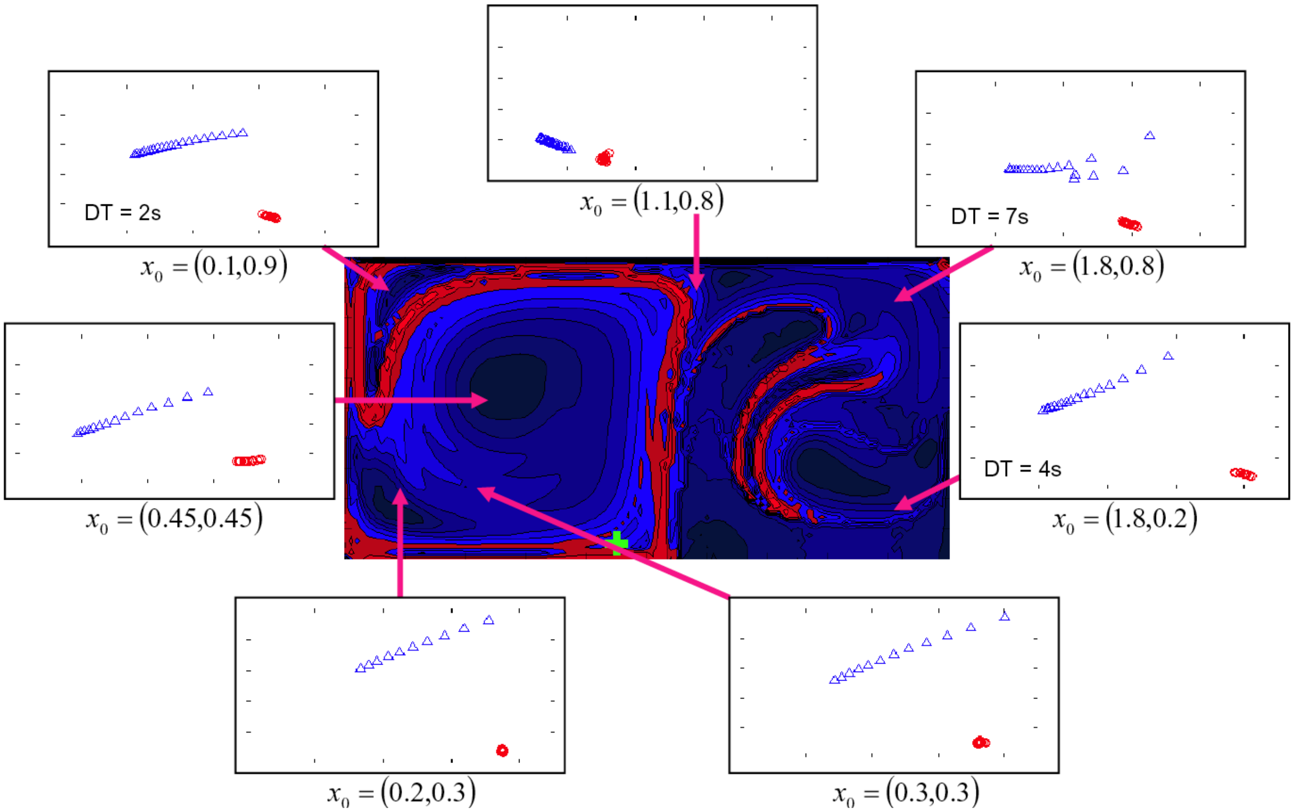

Fig. 5. This plot shows Control Cost vs. Time curves for different initial position x_0. DT is the dropping time used. For the smaller panels the legend is the same as Fig. 6

there are some deviations because the control is not always able to overtake the velocity field. Increasing the U_{ratio} the trajectory becomes a straight line.

The LCS method stays constant for different value of U_{ratio} because the majority of time is spent in simple advection and so changing the control parameter does not affect the time of convergence or the control cost much. In some cases the LCS is even faster than the NC but this can happen only for small U_{ratio}.

Due to the fact the velocity field is a function of space and time the results obtained for some initial conditions cannot be generalized. We show different plots for different initial conditions x_0 in order to gain an overall point of view. In some cases the LCS method is used by dropping the particle with some time delay in order to optimize the results (by waiting until an LCS came close to the initial condition). This is an important point because the LCS act as separatrices of the motion and if the particle ends up on one side of the LCS versus another, we may lose the advantage of advection, as the particle could be advected far from the target region.

Fig. 5 shows that the slope of the Cost vs. Time curves can be either positive or negative. This depends on the direction of the velocity field encountered during the motion and then on the initial condition.

V. CONCLUSIONS

In this work, we show how LCS, providing useful information about a flow, can help in designing fuel efficient trajectories. When the amount of control is limited and the time is not the main constraint, the use of quasi-passive strategies like the LCS method proposed here can provide a fuel-optimal trajectory generation strategy. The explicit incorporation of LCS locations could be used as a more efficient way to initialize optimal control problems where the choice of a good first guess can drastically reduce the optimal control algorithm time.

In the more complex flows which we are currently considering, there is more than one identifiable LCS. This leads to the need for advanced tools able to find the fuel-optimal trajectories by using more than one LCS. Another aspect to be considered in future work is the possibility of having

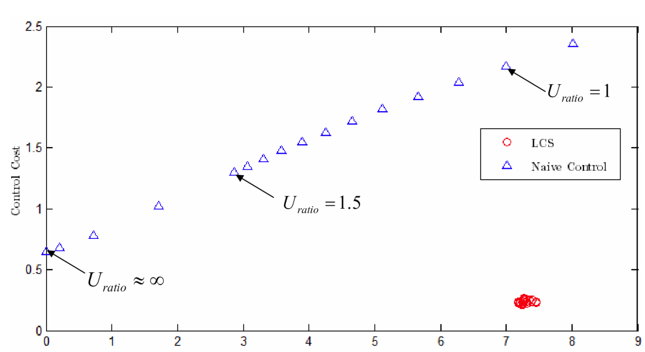

Fig. 6. Control Cost vs. Time

multiple and perhaps moving targets. Further investigation is also needed to optimize the control action ($U_{max} \neq \mathrm{constant}$), to take into account the interactions of the vehicle with the flow. Finally, for many applications, for instance aerial transport, it is important to extend these results to flows in 3D where the LCS then become 2D sheets instead of 1D curves.

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