THE EARTH-MOON LOW-ENERGY TRANSFER IN THE 4-BODY PROBLEM

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A low energy transfer from the Earth to the Moon is proposed in the context of the 4-body Problem. We propose a new model by regarding the Sun-Earth-Moon-Spacecraft (S/C) 4-body system as the coupled system of the Sun-perturbed 3-body system and the Moon-perturbed 3-body system. In particular, we clarify the tube structures of invariant manifolds of the 4-body Problem by investigating the Lagrangian coherent structures of such a coupled 3-body system with perturbations. Lastly, we construct a low-energy transfer trajectory from the Earth to the Moon by patching two trajectories obtained from the perturbed systems at a Poincare section. We develop an optimal trajectory by minimizing the Delta-v at the Poincaré section.

INTRODUCTION

For design of space missions, much effort has been dedicated to construct a transfer trajectory of a spacecraft under the gravitational forces dues to many bodies. The so-called *patched conic approximation* has been often used for interplanetary transfers, in which an approximation of patching two-body problems is made to simplify the trajectory analysis. However, the two-body approximation apparently does not provide sufficiently accurate results, in particular, from the viewpoint of low energy transfer design. So, more efficient and accurate methods have been developed by employing dynamical systems theory. Belbruno and Miller(1993)[1] proposed a new idea for the Earth-Moon transfer design in the context of the restricted 4-body problem for the Earth-Moon-S/C system with the Sun perturbation by introducing the notion of the *weak stability boundary*. The idea was implemented in the Japanese "Hiten" Mission in 1991. Later, another Hiten-like trajectory was shown by Koon et al.(2001)[2] in the context of the *coupled planar restricted circular 3-body problem*, in which the Sun-Earth-Moon-S/C 4-body system is regarded as a coupled system of the Earth-Moon-S/C 3-body system and the Sun-Earth-S/C 3-body system. In particular, they employed the so-called *tube dynamics* in order to systematically construct the required trajectory from the Earth to the Moon by using the characteristics of invariant manifolds.

In this paper, we will show a new idea for the Earth-Moon transfer design by extending the coupled 3-body system to the 3-body system with perturbations due to the Sun or the Moon, in which the Sun-Earth-Moon-S/C 4-body system may be regarded as a coupled system of the Sun-Earth-S/C 3-body system with the Moon's perturbation and the Earth-Moon-S/C 3-body system with the Sun's perturbation.

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Planar restricted circular 3-body problem and tube dynamics

In this section, let us first make a brief review on the dynamics of the planar restricted circular 3-body problem (PRC3BP), in which a spacecraft is subjected by the gravity of two planets. It is assumed that the two planets may move along each circle with a constant angular velocity around the common mass center and also that the spacecraft with a negligible mass moves in the plane of the circles as shown in Figure 1. We assume that the collision points between the spacecraft and the planets are removed. Let m_1 and m_2 $(m_2 < m_1)$ be the masses of the primary and secondary planets respectively. Choose the unit of mass as $m_1 + m_2$, the unit of length as the distance between the primary and secondary planets as well as the unit of time so that the planets period becomes 2π, and the system is to be nondimensional, where the constant of gravitation G can be set to 1. Let $q = (x, y)^T \in Q = \mathbb{R}^2$ be the position of the spacecraft in the rotating frame with the planets and denote by $\dot{q} = dq/dt = (v_x, v_y)^T \in T_q Q \cong \mathbb{R}^2$ the velocity with respect to the nondimensional time t. In the above, $Q = \mathbb{R}^2$ denotes the configuration space and $TQ = \mathbb{R}^2 \times \mathbb{R}^2$ the velocity phase space (the tangent bundle) of Q. Introducing the mass parameter by $\mu = m_2/(m_1 + m_2)$, the equation of motion in the rotating frame[3] can be described by

$$\ddot{q} - 2\tilde{\Omega}\,\dot{q} - q = -\frac{(1-\mu)}{|q-q_1|^3}(q-q_1) - \frac{\mu}{|q-q_2|^3}(q-q_2), \qquad (1)$$

where

$$\tilde{\Omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In the above, $q_1 = (-\mu, 0)^T$ and $q_2 = (1-\mu, 0)^T$ indicate the positions of the primary and secondary planets, respectively. The total energy is given by the sum of the kinetic energy and the effective potential as

$$E(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 - \frac{1}{2}|q|^2 - \frac{1-\mu}{|q-q_1|} - \frac{\mu}{|q-q_2|},$$
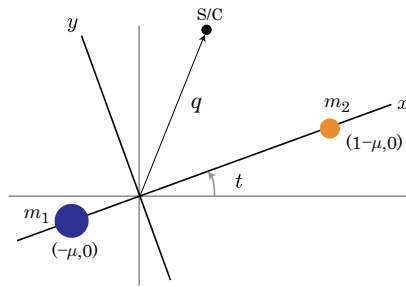
which preserves along the solution curve of the PRC3BP.
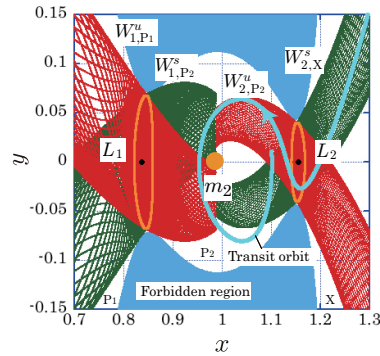

Figure 1: PRC3BP

Figure 2: Flows around the secondary planet

It follows from equation (1) that there exist the three Lagrange points (L_1, L_2, L_3) on the x axis together with the equilateral triangle points (L_4, L_5). Fixing the energy E to some constant value

E_0, we can define an energy surface $\mathcal{E} \subset TQ$ by

$$\mathcal{E}(\mu, E_0) = \{w = (x, y, v_x, v_y) \in TQ \mid E(x, y, v_x, v_y) = E_0\}. \tag{2}$$

Let $\tau : TQ \rightarrow Q$, and one can define Hill's region, in which the spacecraft can energetically move around, by projecting the energy surface $\mathcal{E}(\mu, E_0)$ onto Q as $\tau(\mathcal{E}(\mu, E_0)) \subset Q$. The forbidden region is thus defined as the region without Hill's region, in which the spacecraft is not energetically permitted to move around. In this paper, we choose the energy slightly greater than the energy at L_2 so that the neck regions exist around L_1 and L_2, as shown in Figure 2.

The collinear Lagrange points (L_1, L_2, L_3) are known as the unstable *saddle× center* equilibrium points. Then, the unstable periodic orbits, called the Lyapunov orbits, exist around L_1 and L_2. One can obtain the stable and unstable manifolds, associated with the Lyapunov orbits, which are homeomorphic to $S^1 \times \mathbb{R}$; hence, they are called *tubes*. Hill's region can be divided into three regions by the y-axes including L_1 and L_2, namely, P_1 region including the primary planet, P_2 region including secondary planet and X region without P_1 and P_2 regions, and then we denote by $W_{i,A}^s$ the stable manifolds which asymptotically approach to the Lyapunov orbit around $L_i, (i = 1, 2)$ from $A(= P_1, P_2, X)$ region, while we denote by $W_{i,A}^u$ the unstable manifolds which leave from a Lyapunov orbit around $L_i, (i = 1, 2)$ toward another region $A(= P_1, P_2, X)$. Hence, the tubes separate orbits into *transit* and *non-transit orbits*;[4] namely, an orbit inside of the tubes is to be a transit orbit. For example, if a spacecraft is inside of the tube in some region, it is to be transported to another region. On the other hand, an orbit existing outside of the tubes is to be a non-transit orbit, where a spacecraft in a region remains in the same region.

In Figure 2, we show the Lyapunov orbits colored in orange and the stable and unstable manifolds in green and red, respectively. The transit orbit from X region to P_2 region is depicted in color of cyan in Figure 2.

Earth-Moon transfer in the coupled PRC3BP

Let us consider an Earth-Moon transfer by the coupled PRC3BP following Koon et al.[2] The motion of the spacecraft under the influence of the gravity of the Sun, the Earth and the Moon may be approximately modeled by two distinct PRC3B systems; namely, the Sun-Earth-S/C system with the mass parameter $\mu_S = m_E/(m_S + m_E) = 3.02319 \times 10^{-6}$ and the Earth-Moon-S/C system with the mass parameter $\mu_M = m_M/(m_E + m_M) = 1.21536 \times 10^{-2}$. In this paper, the expression for some quantity or set A is given by local coordinates of the Earth-Moon (E-M) rotating frame, while $\bar{A}$ is expressed by local coordinates of the Sun-Earth (S-E) rotating frame. Let $\bar{w} = (\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) \in TQ$ denote the position and velocity of the spacecraft in the S-E rotating frame.

It follows from equation (2) that the energy surface in the S-E rotating frame is given by

$$\bar{\mathcal{E}}(\mu_S, \bar{E}_{\bar{t}_0}^{SE}) = \{\bar{w} = (\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) \in TQ \mid \bar{E}^{SE}(\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) = \bar{E}_{\bar{t}_0}^{SE}\},$$

where $\bar{E}_{\bar{t}_0}^{SE}$ denotes a given energy of the Sun-Earth-S/C system. Then, define a subspace $\bar{\mathcal{U}}(\mu_S, \bar{E}_{\bar{t}_0}^{SE})$ of the energy surface $\bar{\mathcal{E}}(\mu_S, \bar{E}_{\bar{t}_0}^{SE}) \subset TQ$ by

$$\bar{\mathcal{U}}(\mu_S, \bar{E}_{\bar{t}_0}^{SE}) = \{\bar{w} = (\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) \in \bar{\mathcal{E}}(\mu_S, \bar{E}_{\bar{t}_0}^{SE}) \mid \bar{x} = 1 - \mu_S, \bar{y} > 0, \bar{v}_{\bar{x}} < 0\},$$

where we will show how two different trajectories associated to the Sun-Earth-S/C and Earth-Moon-S/C systems are to be connected at a *patch point* via a Delta-V. Denoting by $(x, y, v_x, v_y) \in TQ$ the

local coordinate in the E-M rotating frame and by also $E_{t_0}^{EM}$ a given energy of the Earth-Moon-S/C system, the energy surface is defined as

$$\mathcal{E}(\mu_{\mathrm{M}}, E_{t_0}^{EM}) = \{w = (x, y, v_x, v_y) \in TQ \mid E^{EM}(x, y, v_x, v_y) = E_{t_0}^{EM}\}.$$

Define a subspace of the energy surface $\mathcal{E}(\mu_{\mathrm{M}}, E_{t_0}^{EM})$ by

$$\mathcal{U}(\mu_{\mathrm{M}}, E_{t_0}^{EM}) = \{w = (x, y, v_x, v_y) \in \mathcal{E}(\mu_{\mathrm{M}}, E_{t_0}^{EM}) \mid x = 0, y > 0, v_x > 0\}.$$

For $w^{EM} \in \mathcal{E}(\mu_{\mathrm{M}}, E_{t_0}^{EM}), \bar{w}^{SE} \in \bar{\mathcal{E}}(\mu_{\mathrm{S}}, \bar{E}_{\bar{t}_0}^{SE})$, we can define the *base patch point* $q = (x, y) = \tau(w^{EM})$ and $\bar{q} = (\bar{x}, \bar{y}) = \tau(\bar{w}^{SE})$ so that they coincide with each other through a coordinate transformation $\varphi : V \times I \to W \times I, (q, t) \mapsto (\bar{q}, \bar{t}) = \varphi(q, t)$ as $(\bar{x}, \bar{y}, \bar{t}) = \varphi(x, y, t)$, where $V, W \subset Q = \mathbb{R}^2$ and $I \subset \mathbb{R}$. Then, the coordinate transformation for the velocity phase space is given by $\tilde{\varphi} : TV \times I \to TW \times I, (w, t) \mapsto (\bar{w}, \bar{t})$. On the other hand, the fiber components of velocities $(\bar{v}_{\bar{x}}, \bar{v}_{\bar{y}})$ at $\bar{t}_0$ and (v_x, v_y) at t_0 take different values in general, because they take different energies $E_{t_0}^{EM}$ and $\bar{E}_{\bar{t}_0}^{SE}$ at the patch point. Here, we assume that $\bar{v}_{\bar{y}}$ and v_y coincide with each other. We illustrate the unstable and stable manifolds on $\bar{\mathcal{U}}(\mu_{\mathrm{S}}, \bar{E}_{\bar{t}_0}^{SE})$ and $\tilde{\varphi}(\mathcal{U}(\mu_{\mathrm{M}}, E_{t_0}^{EM}))$ respectively in Figure 3. The patch point is so chosen that it satisfies the above conditions, while there is a gap between v_x and $\bar{v}_{\bar{x}}$, which will be corrected by a maneuver. Furthermore, the patch points satisfy the conditions so that $\bar{w}^{SE}$ is located outside of the unstable manifolds $\bar{W}_{2,\mathrm{E}}^u$ and w^{EM} is inside of the stable manifolds $W_{2,\mathrm{X}}^s$, as shown in Figure 3. In this setting, we can construct a trajectory near from the Earth to a patch point $\bar{w}^{SE}$ in the Sun-Earth-S/C system by using a non-transit orbit outside of the unstable manifolds $\bar{W}_{2,\mathrm{E}}^u$ via the backward numerical integration, while a trajectory from the patch point w^{EM} to the vicinity of the Moon is developed by a transit orbit inside of the stable manifolds $W_{2,\mathrm{X}}^s$ via the forward numerical integration in the Earth-Moon-S/C system. Hence, the Earth-Moon transfer can be numerically obtained by the backward and forward integrations from the patch point. The obtained Earth-Moon transfer in the S-E (Sun-Earth) rotating frame is shown in Figure 4. Note that a correction maneuver $\Delta V = 0.098$ [km/s] is required in the direction of $\bar{v}_{\bar{x}}$ to connect the trajectories at the patch point.
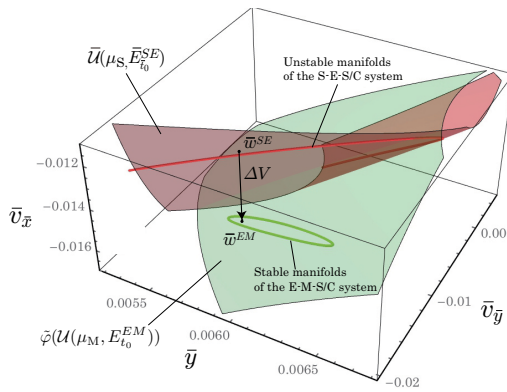

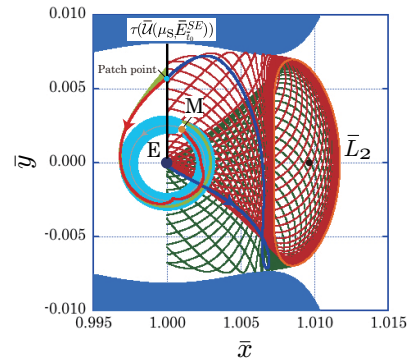**Figure 3:** Invariant manifolds and patch points

Figure 4: Earth-Moon transfer in the coupled PRC3BP in the S-E rotating frame

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The main ideas of our approach

We review our approach to the Earth-Moon transfer design. Our approach is a novel idea which is an extension of the approach based on the coupled PRC3BP to the case with perturbations. In the following sections, we will first regard the Sun-Earth-Moon-S/C 4-body system as a coupled 3-body system with perturbations due to the Sun or the Moon. Namely, the restricted 4-body system can be modeled as a coupled system of the Sun-Earth-S/C 3-body system with the Moon's perturbation (we shall call the *Moon-perturbed system*) and the Earth-Moon-S/C 3-body system with Sun's perturbation (we shall call the *Sun-perturbed system*). Note that the coupled system with perturbations, namely, the restricted 4-body system is a non-conservative system, where the total energy is not preserved along the solution curve anymore. Nevertheless, we will show that we can detect to use the characteristic of the stable and unstable manifolds of the coupled system with perturbations by making use of the Lagrangian coherent structures (LCS), where the LCS is shown to be obtained as the ridges in the Finite time Lyapunov exponent (FTLE) field of the two perturbed systems on a Poincaré section. Then we will obtain the tube structures of the invariant manifolds and also show that they may separate trajectories into the transit and non-transit orbits. The non-transit orbit will be constructed in the Moon-perturbed system as the departure trajectory from near the Earth, and the transit orbit as the arrival trajectory close to the Moon in the Sun-perturbed system. In order to develop the Earth-Moon transfer, we will need to properly choose a patch point on the Poincaré section so that the point is to be outside of the unstable manifolds of the Moon-perturbed system and simultaneously to be inside of the stable manifolds of the Sun-perturbed system. We will finally show how to make an optical design of the Earth-Moon transfer with a zero-maneuver by patching the trajectories.

THE COUPLED 3-BODY SYSTEM WITH PERTURBATIONS

The Bicircular model for the planar restricted 4-body problem

We consider the Bicircular model[5,6] for the Restricted Sun-Earth-Moon-S/C 4-body system as illustrated in Figure 5, where the Sun and the barycenter of the Earth and the Moon (Earth-Moon barycenter) rotate along the circular orbits around the mass center CM of the whole system. The distance between the Sun and the Earth-Moon barycenter is given by $a_{\mathrm{S}}(= 1.49598 \times 10^8 \text{ km})$ and the angular velocity of the Sun and the barycenter is denoted by $\omega_{\mathrm{S}}(= 1.99640 \times 10^{-7} \text{ 1/s})$. The Earth and the Moon rotate along the circular orbits around their barycenter with the angular velocity $\omega_{\mathrm{M}}(= 2.66498 \times 10^{-6} \text{ 1/s})$, and the distance between the planets is $a_{\mathrm{M}}(= 3.84400 \times 10^5 \text{ km})$. The masses of the Sun, the Earth and the Moon are denoted by $m_{\mathrm{S}}(= 1.99976 \times 10^{30} \text{ kg}), m_{\mathrm{E}}(= 5.97219 \times 10^{24} \text{ kg})$ and $m_{\mathrm{M}}(= 7.34767 \times 10^{22} \text{ kg})$ respectively. We assume that the spacecraft and the planets move on the same plane.

We will show that the Bicircular model can be regarded as the coupled 3-body system with perturbations by splitting the motion of the spacecraft into mathematical models described in two different rotating frames.

The Moon-perturbed system and the S-E rotating frame

We normalize system quantities by choosing the unit mass as $m_{\mathrm{S}} + m_{\mathrm{E}} + m_{\mathrm{M}}$, unit length as a_{S} and unit time as $T_{\mathrm{S}} = 2\pi/\omega_{\mathrm{S}}$ such that the constant of gravitation G is set to be 1. We define the mass parameters by $\mu_{\mathrm{S}} = (m_{\mathrm{E}} + m_{\mathrm{M}})/(m_{\mathrm{S}} + m_{\mathrm{E}} + m_{\mathrm{M}}) = 3.02319 \times 10^{-6}$ and $\mu_{\mathrm{M}} = m_{\mathrm{M}}/(m_{\mathrm{E}} + m_{\mathrm{M}}) = 1.21536 \times 10^{-2}$. By this normalization, the distance between the Earth
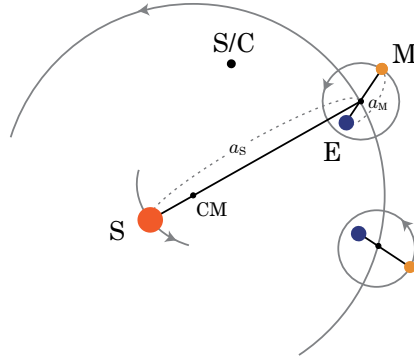
Figure 5: Bicircular model

and the Moon is to be $\alpha_M = a_M/a_S$. The angular velocity of the system of the Earth and the Moon is to be $\varpi_M = \omega_M/\omega_S$. Denoting by $\bar{t}$ the normalized time, then the angle with respect to the line of the Sun and the Earth-Moon barycenter is given by $\bar{\theta}_M = (\varpi_M - 1)\bar{t} + \bar{\theta}_{M0}$, where $\bar{\theta}_{M0}$ indicates an initial value for $\bar{\theta}_M(\bar{t})$. As shown in Figure 6, we set a local coordinate system that rotates with the Sun and the Earth-Moon barycenter, which we shall call the S-E rotating frame.

Figure 6: Bicircular model in the S-E rotating frame

Denoting by $\bar{q} = (\bar{x}, \bar{y})^T \in Q$ the position of the spacecraft in the S-E rotating frame and by $\bar{q}' = d\bar{q}/d\bar{t} = (\bar{v}_{\bar{x}}, \bar{v}_{\bar{y}})^T \in T_{\bar{q}}Q$ the velocity, the equation of motion in the S-E rotating frame is given by

$$\bar{q}'' - 2\tilde{\Omega}\,\bar{q}' - \bar{q} = -\frac{(1-\mu_S)}{|\bar{q}-\bar{q}_S|^3}(\bar{q}-\bar{q}_S) - \frac{\mu_S(1-\mu_M)}{|\bar{q}-\bar{q}_E|^3}(\bar{q}-\bar{q}_E) - \frac{\mu_S\mu_M}{|\bar{q}-\bar{q}_M|^3}(\bar{q}-\bar{q}_M), \qquad (3)$$

where the positions of the Sun, the Earth and the Moon are indicated by

$$\bar{q}_S = (-\mu_S, 0)^T,$$
$$\bar{q}_E = ((1-\mu_S) - \alpha_M\mu_M \cos(\bar{\theta}_M), \ -\alpha_M\mu_M \sin(\bar{\theta}_M))^T,$$
$$\bar{q}_M = ((1-\mu_S) + \alpha_M(1-\mu_M) \cos(\bar{\theta}_M), \ \alpha_M(1-\mu_M) \sin(\bar{\theta}_M))^T,$$

respectively. The energy in the S-E rotating frame is defined by

$$\bar{E}^{SE} = \frac{1}{2}|\bar{q}'|^2 - \frac{1}{2}|\bar{q}|^2 - \frac{(1-\mu_S)}{|\bar{q}-\bar{q}_S|} - \frac{\mu_S(1-\mu_M)}{|\bar{q}-\bar{q}_E|} - \frac{\mu_S\mu_M}{|\bar{q}-\bar{q}_M|}.$$

Note that this energy is *not* conserved along a solution curve since the system is non-autonomous.

If $\mu_M = 0$, that is, the Moon is neglected, then equation (3) coincides with the equation of motion of the Sun-Earth-S/C system. Therefore, the Bicircular model can be considered as the Sun-Earth-S/C system with the Moon's perturbation. In this paper, we call the Bicircular model in the S-E rotating frame as the *Moon-perturbed system*.

The Sun-perturbed system and the E-M rotating frame

Now we introduce another description of the Bicircular model by using the E-M rotating frame, i.e., the local coordinate system rotating with the Earth and the Moon as shown in Figure 7. Choosing the unit mass as $m_E + m_M$, unit length as a_M and unit time as $T_M = 2\pi/\omega_M$, the constant of gravitation G is to be 1. Then, the distance between the Sun and the Earth-Moon barycenter is to be $\alpha_S = a_S/a_M$ and the angular velocity of the system of the Sun and the barycenter can be described as $\varpi_S = \omega_S/\omega_M$.
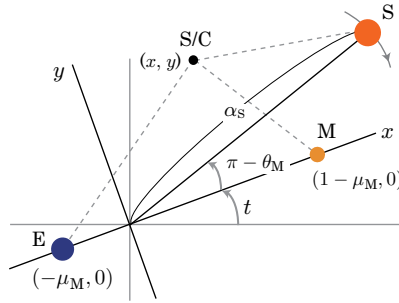

Figure 7: Bicircular model in the E-M rotating frame

Let $t \in I = [t_0 - T, t_0 + T] \subset \mathbb{R}$ be the time in the E-M rotating frame, where t_0 denotes an origin of the time interval which is usually set to 0 and $T > 0$ a certain time interval. Then the relative angle between the planets is given by $\theta_M(t) = (1 - \varpi_S)t + \theta_{M0}$. Denoting by $q = (x, y)^T \in Q$ the potion of the spacecraft in the E-M rotating frame and $\dot{q} = dq/dt = (v_x, v_y)^T \in T_q Q$, the equation of motion in the E-M rotating frame is obtained as

$$\ddot{q} - 2\tilde{\Omega}\dot{q} - q = -\frac{1 - \mu_M}{|q - q_E|^3}(q - q_E) - \frac{\mu_M}{|q - q_M|^3}(q - q_M) - \frac{1 - \mu_S}{\mu_S|q - q_S|^3}(q - q_S) - \frac{1 - \mu_S}{\mu_S \alpha_S^3} q_S. \quad (4)$$

In the above equation, q_S, q_E and q_M indicate the position vectors of the Sun, the Earth and the Moon, respectively, given by

$$q_S = (-\alpha_S \cos(\theta_M(t)), \alpha_S \sin(\theta_M(t)))^T,$$
$$q_E = (-\mu_M, 0)^T,$$
$$q_M = (1 - \mu_M, 0)^T.$$

We define the energy in the E-M rotating frame[6] by

$$E^{EM} = \frac{1}{2}|\dot{q}|^2 - \frac{1}{2}|q|^2 - \frac{1 - \mu_M}{|q - q_E|} - \frac{\mu_M}{|q - q_M|} - \frac{1 - \mu_S}{\mu_S|q - q_S|} + \frac{1 - \mu_S}{\mu_S \alpha_S^3}\langle q_S, q\rangle,$$

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which is not conserved along a trajectory.

If we choose $\mu_S = 1$, that is, the Sun is neglected, then equation (4) is to be the equation of motion of the Earth-Moon-S/C system. Thus, we can consider the Bicircular model in the E-M rotating frame as the Earth-Moon-S/C system with the Sun's perturbation, which we shall call the *Sun-perturbed system.*[9]

The coordinate transformation

Now we show the coordinate transformation between the Moon-perturbed system and the Sun-perturbed system. The transformation of time is given by

$$\bar{t} = \frac{\omega_S}{\omega_M} t = \varpi_S t.$$

The transformation of the position vectors is

$$\bar{q} = \bar{q}_B + \frac{a_M}{a_S} C(t) q = \bar{q}_B + \frac{1}{\alpha_S} C(t) q, \tag{5}$$

where $\bar{q}_B = (1 - \mu_S, 0)^T$ denotes the position of the Earth-Moon barycenter in the S-E rotating frame and $C(t)$ is a rotation matrix given by

$$C(t) = \begin{pmatrix} \cos\theta_M(t) & -\sin\theta_M(t) \\ \sin\theta_M(t) & \cos\theta_M(t) \end{pmatrix}.$$

The transformation of the velocity can be obtained from the differentiation of equation(5).

TUBE DYNAMICS IN THE PERTURBED SYSTEMS

Since the 4-body system can be regarded as a 3-body system with perturbations, i.e., the Moon-perturbed system or the Sun-perturbed system, the stable and unstable invariant manifolds of the perturbed 3-body systems can be detected in principle. On the other hand, it is generally difficult to figure out such stable and unstable manifolds since the Moon-perturbed system and the Sun-perturbed system are non-autonomous. Here, we will explore the Lagrangian coherent structures[7,8] to detect the stable and unstable manifolds in the perturbed systems and then we will show the tube structures of the invariant manifolds, which are an extension of the separatrix.

Finite time Lyapunov exponents

Let us review the notions of the finite time Lyapunov exponent and the Lagrangian coherent structure.[7]

Let D be a subset of a phase space $M \subset \mathbb{R}^n$. Consider a time-dependent dynamical system

$$\begin{cases} \dot{\mathbf{x}}(t; t_0, \mathbf{x}_0) = \mathbf{v}(\mathbf{x}(t; t_0, \mathbf{x}_0), t), \\ \mathbf{x}(t_0; t_0, \mathbf{x}_0) = \mathbf{x}_0, \end{cases}$$

where $\mathbf{x}(t; t_0, \mathbf{x}_0)$ is a smooth solution curve starting at an initial point $\mathbf{x}_0 \in D$ at time t_0 and $\mathbf{v}(\mathbf{x}, t)$ is a given time-dependent vector field. Then, the point $\mathbf{x}_0$ moves to another point after a finite time interval T by the flow map:

$$\phi_{t_0}^{t_0+T} : D \to D; \qquad \mathbf{x}_0 \mapsto \phi_{t_0}^{t_0+T}(\mathbf{x}_0) = \mathbf{x}(T; t_0, \mathbf{x}_0).$$

The Finite Time Lyapunov Exponent (FTLE) denotes a finite time average of the maximum expansion or contraction rate for a pair of particles neighboring in the initial time while advecting by the flow map. Now, suppose a perturbed point $\mathbf{y} = \mathbf{x} + \delta\mathbf{x}(0)$, where $\delta\mathbf{x}(0)$ is infinitesimal. After the time interval T, the perturbation is to be

$$\delta\mathbf{x}(T) = \phi_{t_0}^{t_0+T}(\mathbf{y}) - \phi_{t_0}^{t_0+T}(\mathbf{x}) = \frac{d\phi_{t_0}^{t_0+T}(\mathbf{x})}{d\mathbf{x}}\delta\mathbf{x}(0) + O(\|\delta\mathbf{x}(0)\|^2).$$

By neglecting the higher order terms $O(\|\delta\mathbf{x}(0)\|^2)$, the magnitude of the perturbation becomes

$$\|\delta\mathbf{x}(T)\| = \sqrt{\langle\delta\mathbf{x}(0), \Delta\delta\mathbf{x}(0)\rangle},$$

where Δ is a symmetric matrix given by

$$\Delta = \left(\frac{d\phi_{t_0}^{t_0+T}(\mathbf{x})}{d\mathbf{x}}\right)^* \frac{d\phi_{t_0}^{t_0+T}(\mathbf{x})}{d\mathbf{x}}.$$

This is a *finite time* version of the *(right) Cauchy-Green tensor*.

The maximum stretching is given when $\delta\mathbf{x}(0)$ is so chosen that it is aligned with the eigenvector of the maximum eigenvalue of Δ, which we denote by $\lambda_{\max}(\Delta)$. Let $\overline{\delta\mathbf{x}}(0)$ be an initial perturbation aligned with the eigenvector, and it follows that

$$\max_{\delta\mathbf{x}(0)} \|\delta\mathbf{x}(T)\| = \sqrt{\lambda_{\max}(\Delta)} \left\|\overline{\delta\mathbf{x}}(0)\right\|,$$

which may be restated by

$$\max_{\delta\mathbf{x}(0)} \|\delta\mathbf{x}(T)\| = e^{\sigma_{t_0}^{t_0+T}(\mathbf{x})|T|} \left\|\overline{\delta\mathbf{x}}(0)\right\|.$$

The *Finite time Lyapunov exponent (FTLE) field* $\sigma_{t_0}^{t_0+T} : D \subset M \to \mathbb{R}$ associated with a finite time T is defined by

$$\sigma_{t_0}^{t_0+T}(\mathbf{x}) = \frac{1}{|T|} \ln\sqrt{\lambda_{\max}(\Delta)}.$$

In this paper, we choose $t_0 = 0$ to compute the FTLE field for the 4-body system.

In order to numerically compute the FTLE field, we advect a regularly spaced rectilinear grid of tracers forward in time by T by using the Runge-Kutta-Fehlberg integrator with 8th order, in which we discretize the matrix $\dfrac{d\phi_{t_0}^{t_0+T}(\mathbf{x})}{d\mathbf{x}}$ as

$$\frac{\partial[\phi_{t_0}^{t_0+T}(\mathbf{x})]_i}{\partial x_j} \approx \frac{[\phi_{t_0}^{t_0+T}(\mathbf{x} + \Delta\mathbf{x}_j)]_i - [\phi_{t_0}^{t_0+T}(\mathbf{x} - \Delta\mathbf{x}_j)]_i}{2\Delta\mathbf{x}_j},$$

where x_i and x_j are components of $\mathbf{x}$ by the central difference approximation.

Lagrangian coherent structures

Recall the definition of the *Lagrangian coherent structure* as a ridge of the FTLE field,[7] where the LCS is defined by the *second-derivative ridge* of an FTLE field σ^T, namely, given by the Hessian of the FTLE field as

$$\Sigma = \frac{d^2 \sigma^T(\mathbf{x})}{d\mathbf{x}^2}.$$

In the above, we employ the notation σ^T as the FTLE field for brevity. More formally, the second-derivative ridge of σ^T is defined by an injective curve $\mathbf{c} : (a, b) \to D$ that satisfies the following conditions:

The vectors $\mathbf{c}'(s)$ and $\nabla \sigma_{t_0}^T(\mathbf{c}(s))$ are parallel. In addition to the condition, $\Sigma(\mathbf{n}, \mathbf{n}) = \min_{\|\mathbf{u}\|=1} \Sigma(\mathbf{u}, \mathbf{u}) < 0$ is required, where $\mathbf{n}$ is a unit normal vector to the curve $\mathbf{c}(s)$ and Σ is thought of as a bilinear form evaluated at the point $\mathbf{c}(s)$.

A ridge of the backward-time FTLE field, which one can obtain by the negative integration time T, is called the *attracting Lagrangian coherent structure* that corresponds to the time-dependent analogue of the unstable manifold, while a ridge of the forward-time FTLE field with the positive integration time T the *repelling Lagrangian coherent structure* that corresponds to the stable manifold.

FTLE field and LCS in the Moon-perturbed system

Now let us consider how to extract the attracting LCS for the Moon-perturbed system from the FTLE field. To do this, define an instantaneous energy surface at $\bar{t} = \bar{t}_0$ in the S-E rotating frame by

$$\bar{\mathcal{E}}(\mu, \bar{E}_{\bar{t}_0}^{SE}) = \{\bar{w} = (\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) \in TQ \mid \bar{E}^{SE}(\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}, \bar{t}_0) = \bar{E}_{\bar{t}_0}^{SE}\}.$$

In the above, $\mu = (\mu_{\mathrm{S}}, \mu_{\mathrm{M}})$ and $\bar{E}_{\bar{t}_0}^{SE}$ denotes a fixed value for the energy at $\bar{t} = \bar{t}_0$ in the S-E rotating frame, and we set $\bar{E}_{\bar{t}_0}^{SE} = -1.50037$ so that the instantaneous Hill's region has a neck-like region. Define a subspace $\bar{\mathcal{U}} \subset \bar{\mathcal{E}}(\mu, \bar{E}_{\bar{t}_0}^{SE})$ at $\bar{t} = \bar{t}_0$ in the Moon-perturbed system by

$$\bar{\mathcal{U}}(\mu, \bar{E}_{\bar{t}_0}^{SE}) := \{\bar{w} = (\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) \in \bar{\mathcal{E}}(\mu, \bar{E}_{\bar{t}_0}^{SE}) \mid \bar{\theta}_{\mathrm{M0}} = 0, \bar{x} = 1 - \mu_{\mathrm{S}}, \bar{y} > 0, \bar{v}_{\bar{x}} < 0\}.$$

In order to see the FTLE field on the $(\bar{y} - \bar{v}_{\bar{y}})$-plane, we introduce a projection

$$\pi : (\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) \mapsto (\bar{y}, \bar{v}_{\bar{y}})$$

and set 250×2500 grids within $(\bar{y}, \bar{v}_{\bar{y}}) \in [0.002, 0.007] \times [-0.05, 0.02]$ on the Poincaré section $\bar{U}_0 := \pi(\bar{\mathcal{U}})$, and we can compute the FTLE field for the integral time $T = -7$ as illustrated in Figure 8.

Here, let us introduce notations for the i-th intersection of the unstable manifolds $\bar{W}_{2,\mathrm{E}}^u$ and $\bar{W}_{1,\mathrm{E}}^u$ with the subspace $\bar{U}_0$, denoted by $\bar{\Gamma}_{2,\mathrm{E}}^{i,u}$ and $\bar{\Gamma}_{1,\mathrm{E}}^{i,u}$. So, $\bar{\Gamma}_{2,\mathrm{E}}^{1,u}$ denotes the subset $\bar{W}_{2,\mathrm{E}}^u \cap \bar{U}_0$ for the first intersection of $\bar{W}_{2,\mathrm{E}}^u$ and $\bar{U}_0$, which is the unstable manifolds toward E region associated to the Lyapunov orbit $\bar{L}_2$ on $\bar{U}_0$. Here, we use the same notations $\bar{W}_{2,\mathrm{E}}^u$ and $\bar{\Gamma}_{2,\mathrm{E}}^{1,u}$ to describe the unstable manifolds and the subset $\bar{W}_{2,\mathrm{E}}^u \cap \bar{U}_0$ as those for the Sun-Earth-S/C 3-body system.[8]

To detect the attracting LCS corresponding to $\bar{\Gamma}_{2,\mathrm{E}}^{1,u}$ from the FTLE field for the Moon-perturbed system, let us introduce the line $l_{\bar{\varphi}}$ on $\bar{U}_0$ at $\bar{\varphi} \in [-\pi/2, \pi/2]$ defined by

$$l_{\bar{\varphi}} = \{(\bar{y}, \bar{v}_{\bar{y}}) \in \bar{U}_0 \mid \bar{y} = \bar{y}_c + r\cos\bar{\varphi}, \bar{v}_{\bar{y}} = \bar{v}_{\bar{y}_c} + r\sin\bar{\varphi}, r \in [0, r_{\max}] \subset \mathbb{R}\},$$
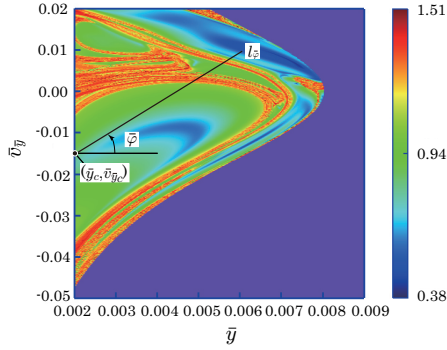
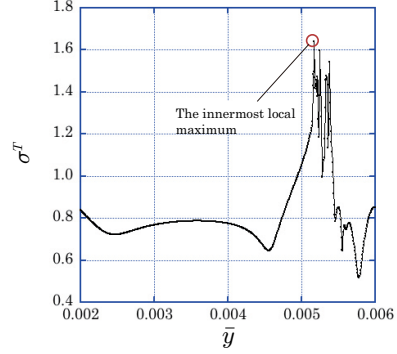**Figure 8:** FTLE field ($T = -7$)
in the Moon-perturbed system

Figure 9: FTLE on the line l_0 ($T = -7$)
in the Moon-perturbed system

where $r_{\max} = \sqrt{(0.006 \cos \bar{\varphi})^2 + (0.025 \sin \bar{\varphi})^2}$ and $(\bar{y}_c, \bar{v}_{\bar{y}c}) = (0.002, -0.015) \in \bar{U}_0$. In order to extract the LCS on $\bar{U}_0$, we compute the values of the FTLE on the line as illustrated in Figure 9.

The attracting LCS can be obtained as the *innermost local maximum* of the FTLE as in Figure 10. Set $\bar{w}^l = (\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) = (0.999997, 0.00516195, -0.0138839, -0, 15)$ as the initial point at $\bar{t}_0$. So we obtain a trajectory that approaches to Lyapunov-like orbit around $\bar{L}_2$ by backward integration for $\bar{w}^l$ as shown in Figure 11.

Figure 10: Attracting LCS on $\bar{U}_0$ ($T = -7$)
in the Moon-perturbed system

Figure 11: Transit and non-transit orbits
in the Moon-perturbed system

We also illustrate the cases in which the initial points are

$$\bar{w}^t = (\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) = (0.999997, 0.0048, -0.0230564, -0, 15),$$
$$\bar{w}^n = (\bar{x}, \bar{y}, \bar{v}_{\bar{x}}, \bar{v}_{\bar{y}}) = (0.999997, 0.00535, -0.00544582, -0.015),$$

where $\bar{w}^t$ and $\bar{w}^n$ are located inside and outside of the LCS, respectively. The obtained trajectories are to be transit and non-transit orbits associated with $\bar{w}^t$ and $\bar{w}^n$ respectively as shown in Figure 11.

Similary, we compute the attracting LCS for the cases with the instantaneous energy at $\bar{t} = \bar{t}_0$:

$$\bar{E}^{SE}_{\bar{t}_0} = -1.50040, -1.50039, -1.50038, -1.50037.$$
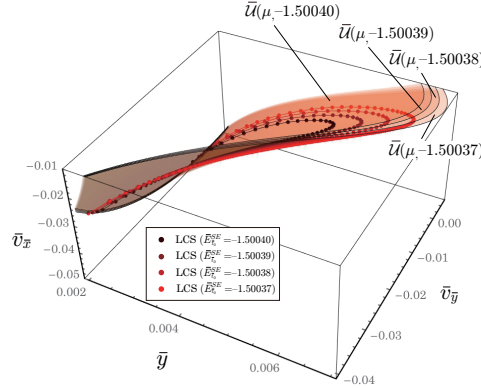
Figure 12: Attracting LCS on $\bar{\mathcal{U}}(\mu, \bar{E}^{SE}_{\bar{t}_0})$ $(T = -7)$
in the Moon-perturbed system

Note that fixing $(\bar{x}, \bar{y}, \bar{v}_{\bar{y}})$ under $\bar{E}^{SE} = \bar{E}^{SE}_{\bar{t}_0}$ corresponds to the choice of the velocity $\bar{v}_{\bar{x}}$ at the patch point $\bar{w}$ at $\bar{t} = \bar{t}_0$. We illustrate the set of the LCSs associate with the different energies on $\bar{\mathcal{U}}(\mu, \bar{E}^{SE}_{\bar{t}_0})$ in Figure 12.

FTLE field and LCS in the Sun-perturbed system

Let us compute the repelling LCS corresponding to the stable manifolds $W^s_{2,\mathrm{X}}$ in the Sun-perturbed system. We use the similar notation $W^s_{2,\mathrm{X}}$ for the stable manifolds in the Sun-perturbed system as in the Moon-perturbed system and we make all the computations by using local coordinates $(x, y, v_x, v_y, t) \in TQ \times \mathbb{R}$ in the E-M rotating frame. First, define an instantaneous energy surface $\mathcal{E}(\mu, E^{EM}_{t_0}) \subset TQ$ at $t = t_0$ in the E-M rotating frame by

$$\mathcal{E}(\mu, E^{EM}_{t_0}) = \{w = (x, y, v_x, v_y) \in TQ \mid E^{EM}(x, y, v_x, v_y, t_0) = E^{EM}_{t_0}\},$$

where $E^{EM}_{t_0}$ denotes a fixed value for the energy at $t = t_0$ in the E-M rotating frame, which is so chosen that the Hill's region at $t = t_0$ has a neck-like region. Define a subspace $\mathcal{U} \subset \mathcal{E}(\mu, E^{EM}_{t_0})$ at $t = t_0$ in the Sun-perturbed system by

$$\mathcal{U}(\mu, E^{EM}_{t_0}) := \{w = (x, y, v_x, v_y) \in \mathcal{E}(\mu, E^{EM}_{t_0}) \mid \theta_{\mathrm{M0}} = 0, x = 0, y > 0, v_x > 0\}.$$

Setting 500×250 grids within $(y, v_y) \in [1.9, 2.7] \times [-0.55, -0.15]$ on the Poincaré section $U_0 := \pi(\mathcal{U})$, the FTLE field on U_0 for $E^{EM}_{t_0} = -851.495$ and $T = 15$ is illustrated in Figure 13. As in the preceding section, we set the line

$$l_\varphi = \{(y, v_y) \in U_0 \mid y = y_c + r\cos\varphi, v_y = v_{yc} + r\cos\varphi, r \in [0, r_{\max}]\},$$

where $(y_c, v_{yc}) = (2.3, -0.36) \in U_0$ and $r_{\max} = 0.4$. The repelling LCS on U_0 corresponding to the subset $\Gamma^{1,s}_{2,\mathrm{X}} = W^s_{2,\mathrm{X}} \cap U_0$ can be computed as the *outermost local maximum* of the FTLE on the line l_φ. We show the FTLE on l_0 in Figure 14 and the extracted LCS in Figure 13.

Setting the initial points $w^1 = (x, y, v_x, v_y) = (0, 2.23744, 1.62525, -0.36) \in TQ$ on the LCS, the integrated orbit is shown in Figure 16. The orbit seems to be asymptotic to the quasi-periodic
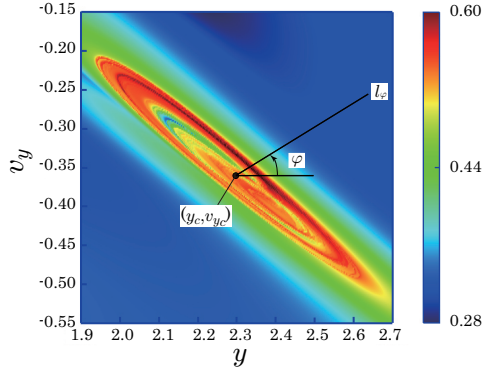
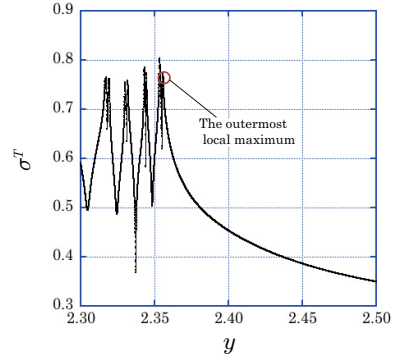**Figure 13:** FTLE field on U_0 $(T = 15)$ in the Sun-perturbed system

Figure 14: FTLE on l_0 $(T = 15)$ in the Sun-perturbed system
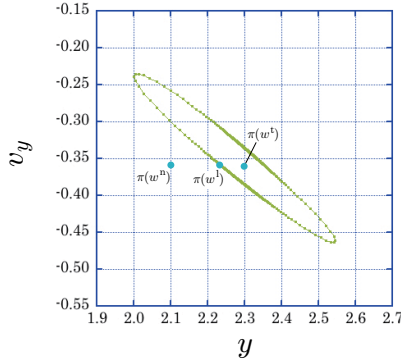

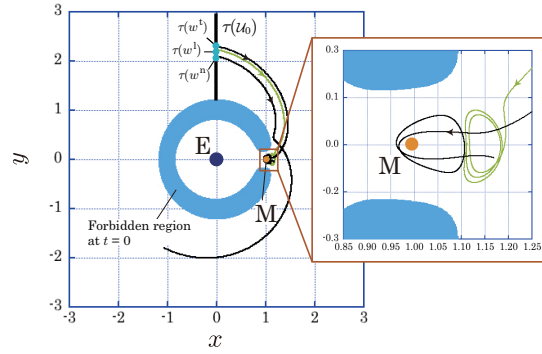**Figure 15:** Repelling LCS on U_0 $(T = 15)$ in the Sun-perturbed system

Figure 16: Transit and non-transit orbits in the Sun-perturbed system

orbit. If we choose the initial points inside and outside of the LCS as

$$w^{\mathrm{t}} = (x, y, v_x, v_y) = (0, 2.3, 1.70278, -0.36),$$
$$w^{\mathrm{n}} = (x, y, v_x, v_y) = (0, 2.1, 1.45154, -0.36),$$

respectively, as in Figure 15, then the orbits are to be transit and non-transit orbits in Figure 16. Thus the LCS plays a role of separatrixes in the Sun-perturbed system.

We also calculate the LCS for the cases of the instantaneous energy at $t = t_0$:

$$E_{t_0}^{EM} = -851.500, -851.495, -851.490, -851.485, -851.480.$$

We illustrate the set of the obtained LCSs on the section $\mathcal{U}(\mu, E_{t_0}^{EM})$ in Figure 17.

EARTH-MOON TRANSFER IN THE COUPLED 3-BODY SYSTEM WITH PERTURBATIONS

Let us construct the Earth-Moon transfer by connecting the trajectories associated with the repelling and attracting LCSs. In Figure 18, we illustrate the unstable manifolds corresponding to the
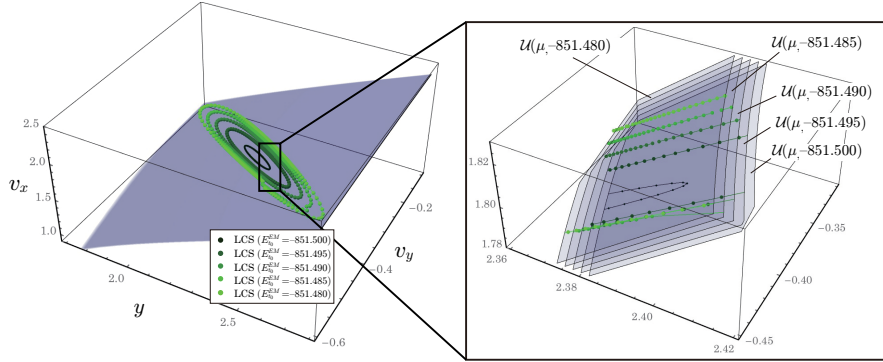
Figure 17: Repelling LCS on $\mathcal{U}(\mu, E_{t_0}^{EM})$ $(T = 15)$
in the Sun-perturbed system

attracting LCS in the Moon-perturbed system on $\bar{\mathcal{U}}(\mu, \bar{E}_{\bar{t}_0}^{SE})$ and the stable manifolds corresponding to the repelling LCS on $\tilde{\varphi}(\mathcal{U}(\mu, E_{t_0}^{EM}))$. Note that the repelling LCS is transformed into the expressions on $\tilde{\varphi}(\mathcal{U}(\mu, E_{t_0}^{EM}))$ in the Sun-perturbed system by equation (5).
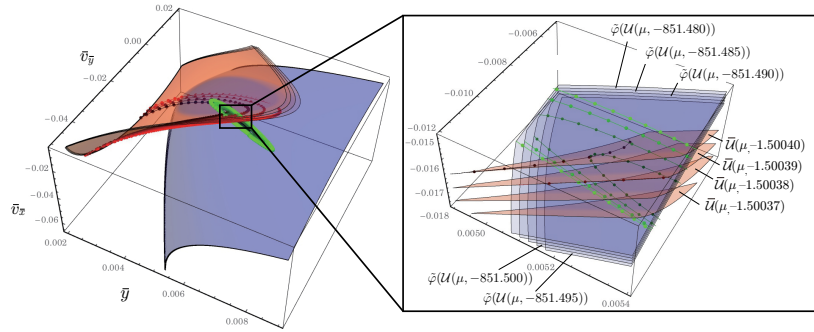

Figure 18: Invariant manifolds on $\bar{\mathcal{U}}(\mu, \bar{E}_{\bar{t}_0}^{SE})$ and $\tilde{\varphi}(\mathcal{U}(\mu, E_{t_0}^{EM}))$

Choose the appropriate patch point to be outside of the unstable manifolds of the Moon-perturbed system and simultaneously be inside of the stable manifolds of the Sun-perturbed system in order to construct the Earth-Moon transfer. It follows from Figure 18 that there exist points satisfying above conditions on an intersection between the set of $\bar{\mathcal{U}}(\mu, \bar{E}_{\bar{t}_0}^{SE})$ and the set of $\tilde{\varphi}(\mathcal{U}(\mu, E_{t_0}^{EM}))$, parametrizing $\bar{E}_{\bar{t}_0}^{SE}$ and $E_{t_0}^{EM}$. Now, we choose the cases of $\bar{E}_{\bar{t}_0}^{SE} = -1.50040$ and $E_{t_0}^{EM} = -851.490$, as in Figure 19, and clearly there exist some candidates for the patch point. So, let us choose the point

$$\bar{w} = (\tilde{x}, \bar{y}, \bar{v}_{\tilde{x}}, \bar{v}_{\bar{y}}) = (1 - \mu_S, 0.00520000, -0.00816162, -0.0167743),$$

as shown in Figure 19. Thus, we can develop the Earth-Moon transfer in the S-E rotating frame by the backward and forward integrations from the patch point, as in Figure 20. We also illustrate the same Earth-Moon transfer in the E-M rotating frame in Figure 21. Note that since the constructed transfer does not require any ΔV in patching the orbits, it is naturally connected near from the Earth to the vicinity of the Moon in the velocity phase space.
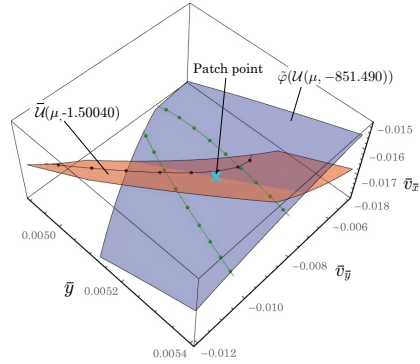
Figure 19: Invariant manifolds and the patch point
on $\bar{\mathcal{U}}(\mu, -1.50040)$ and $\tilde{\varphi}(\mathcal{U}(\mu, -851.490))$
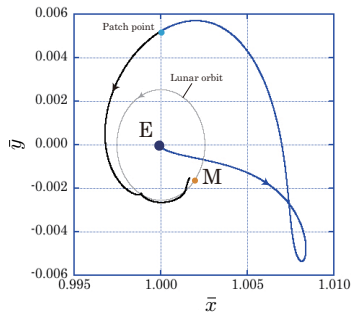

Figure 20: Earth-Moon transfer
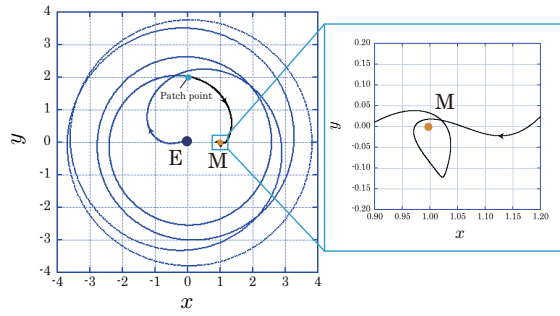in the S-E rotating frame

Figure 21: Earth-Moon transfer in the E-M rotating frame

CONCLUSIONS

We have proposed a novel model for the design of the Earth-Moon transfer of a spacecraft under the gravitational influence due to the Sun. The model is an extension of the coupled 3-body system;[2] namely, we regard the restricted planar 4-body (Bicircular) model of the Sun-Earth-Moon-S/C system as a coupled system of the Sun-Earth-S/C 3-body system with the Moon's perturbation (the Moon-perturbed system) and the Earth-Moon-S/C 3-body system with the Sun's perturbation (the Sun-perturbed system). Note that this model is exactly equivalent with the Bicircular model. We have shown how the Lagrangian coherent structures of the Sun-perturbed and Moon-perturbed systems can be detected as the ridges of the FTLE fields, and then we have also shown the attracting LCS in the Moon-perturbed system and the repelling LCS in the Sun-perturbed system on a Poincaré section by the backward and forward numerical integrations. By using these LCSs, we have shown that the optimal Earth-Moon transfer can be obtained without any the trajectory correction maneuver ($\Delta V = 0$) by choosing the patch point properly.

ACKNOWLEDGMENT

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